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Automorphism groups of Alexander quandles

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ABSTRACT

An Alexander quandle is a module M over $\mathbb{Z}[t, t^{-1}]$ whose quandle operation is defined by $x * y = tx + (1 - t)y$, $x, y \in M$. The automorphism group of a general Alexander quandle, previously unknown, is determined in the present paper.

1. Introduction

A *quandle* is a set X equipped with a binary operation $*$ such that

(i) $x * x = x$ for all $x \in X$;
(ii) for each $y \in X$, the mapping $x \mapsto x * y$ is a bijection from X to X;
(iii) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

A homomorphism from a quandle $(X, *)$ to another quandle $(X', \circ)$ is a mapping $f : X \to Y$ such that $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X$. An isomorphism between two quandles is a bijective homomorphism. The automorphism group of a quandle $(X, *)$ is denoted by $\mathrm{Aut}(X, *)$. Automorphism groups of all quandles of order $\leqslant 5$ have been determined in [5] and the same has been done for quandles of order 6 in [2].

The notion of quandle arises from topology. Associated to each tame knot K in S^3, there is a *fundamental quandle* $Q(K)$ which is a complete invariant of the type of the (unoriented) knot K [7].

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For an extension of this result to a codimension 2 submanifold of a manifold, see [4]. The fundamental quandle of a knot is given by a finite presentation which is encoded in the digram of the knot. It is usually difficult to distinguish two quandles from their presentations alone; a comparison of homomorphisms from the two quandles to simpler and well understood quandles can provide useful information. For additional applications of quandles in topology, see, for example, [1,9,12].

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$. An Alexander quandle is a Λ-module M whose quandle operation is defined by

$$x * y = tx + (1-t)y, \quad x, y \in M. \tag{1.1}$$

When $(1-t)M = M$, the automorphism group of the Alexander quandle $(M, *)$ is known from a result of [10] or Theorem 3.1 of the present paper. In this case, $\mathrm{Aut}(M, *) \cong M \rtimes \mathrm{Aut}_\Lambda(M)$. However, the automorphism group of a general Alexander quandle $(M, *)$ was previously unknown. As we will see, the group $\mathrm{Aut}(M, *)$ is interesting enough to warrant a careful investigation. The purpose of this paper is to determine $\mathrm{Aut}(M, *)$ for a general Alexander quandle $(M, *)$. We begin with a few basic facts about quandles that are relevant to the purpose of the paper (Section 2). It was proved by Nelson [10] that two finite Alexander quandles M and N are isomorphic if and only if $|M| = |N|$ and $(1-t)M \cong (1-t)N$ as Λ-modules. In Section 3, we first extend this criterion to arbitrary Alexander quandles. Afterwards, we determine the automorphisms of a general Alexander quandle $(M, *)$. The structure of the automorphism group $\mathrm{Aut}(M, *)$ is described in Section 4. In the last section, we apply the general result to two examples of Alexander quandles. In the first example, $M = \mathbb{Z}_n$, and $a \in \mathbb{Z}_n^\times$ is chosen to define $tx = ax$, $x \in \mathbb{Z}_n$; in the second example, $M = \mathbb{Z}_p^n$, where p is a prime, and $A \in \mathrm{GL}(n, \mathbb{Z}_p)$ is chosen to define $tx = Ax$, $x \in \mathbb{Z}_p^n$. In these examples, we obtain explicit formulas for the order of $\mathrm{Aut}(M, *)$.

2. A few facts

The following construction can be found in [7, p.42]. Let G be a group and let $\alpha \in \mathrm{Aut}(G)$. For $x, y \in G$, define

$$x * y = \alpha\bigl(xy^{-1}\bigr)y.$$

Then $(G, *)$ is a quandle. For each $a \in G$, define

$$\begin{aligned} r_a : G &\longrightarrow G, \\ x &\longmapsto xa. \end{aligned}$$

Then $r_a \in \mathrm{Aut}(G, *)$. Moreover, $a \mapsto r_{a^{-1}}$ defines an embedding $G \hookrightarrow \mathrm{Aut}(G, *)$.

Proposition 2.1. *Assume the above notation.*

(i) $\mathrm{Aut}(G) \cap \mathrm{Aut}(G, *) = C_{\mathrm{Aut}(G)}(\alpha)$, *where* $C_{\mathrm{Aut}(G)}(\alpha) = \{f \in \mathrm{Aut}(G)\colon\ f\alpha = \alpha f\}$.

(ii) *If* $\{\alpha(x)^{-1}x\colon\ x \in G\} = G$, *then* $\{f \in \mathrm{Aut}(G, *)\colon\ f(1) = 1\} = C_{\mathrm{Aut}(G)}(\alpha)$ *and*

$$\mathrm{Aut}(G, *) \cong G \rtimes C_{\mathrm{Aut}(G)}(\alpha),$$

where the left action of $C_{\mathrm{Aut}(G)}(\alpha)$ *on* G *that defines* $G \rtimes C_{\mathrm{Aut}(G)}(\alpha)$ *is* $f \cdot a = f(a)$, $f \in C_{\mathrm{Aut}(G)}(\alpha)$, $a \in G$.

Proof. First note that $f : G \to G$ is an endomorphism of $(G, *)$ if and only if

$$f\bigl(\alpha(x)\alpha(y)^{-1}y\bigr) = \alpha\bigl(f(x)\bigr)\alpha\bigl(f(y)\bigr)^{-1} f(y) \quad \text{for all } x, y \in G. \tag{2.1}$$

(i) Assume $f \in C_{\mathrm{Aut}(G)}(\alpha)$. Then (2.1) is clearly satisfied, hence $f \in \mathrm{Aut}(G) \cap \mathrm{Aut}(G,*)$.

Next assume $f \in \mathrm{Aut}(G) \cap \mathrm{Aut}(G,*)$. Letting $y = 1$ in (2.1), we have $f(\alpha(x)) = \alpha(f(x))$ for all $x \in G$, namely, $f \in C_{\mathrm{Aut}(G)}(\alpha)$.

(ii) Let $S = \{f \in \mathrm{Aut}(G,*):\ f(1) = 1\}$ and $R = \{r_a:\ a \in G\}$. To prove $S = C_{\mathrm{Aut}(G)}(\alpha)$, by (i) it suffices to show that $S \subset \mathrm{Aut}(G)$. Assume $f \in S$. Letting $y = 1$ and $x = 1$ in (2.1), respectively, we have

$$\begin{aligned} f\bigl(\alpha(x)\bigr) &= \alpha\bigl(f(x)\bigr) && \text{for all } x \in G, \\ f\bigl(\alpha(y)^{-1}y\bigr) &= \alpha\bigl(f(y)\bigr)^{-1}f(y) && \text{for all } y \in G. \end{aligned}$$

Make these two substitutions in the right side of (2.1). We now have

$$f\bigl(\alpha(x)\alpha(y)^{-1}y\bigr) = f\bigl(\alpha(x)\bigr)f\bigl(\alpha(y)^{-1}y\bigr) \quad \text{for all } x, y \in G.$$

Since $\{\alpha(y)^{-1}y:\ y \in G\} = G$, the above equation is equivalent to $f(xy) = f(x)f(y)$, $x, y \in G$. So $f \in \mathrm{Aut}(G)$.

Clearly, $\mathrm{Aut}(G,*) = RS$ and $R \cap S = \{\mathrm{id}\}$. Since $S \subset \mathrm{Aut}(G)$, it is easy to see that $R \lhd RS$. Thus $RS \cong R \rtimes S \cong G \rtimes C_{\mathrm{Aut}(G)}(\alpha)$. $\square$

Remark. In Proposition 2.1(ii), when $|G| < \infty$, the condition that $\{\alpha(x)^{-1}x:\ x \in G\} = G$ means that the automorphism α of G is *fixed-point-free*. Finite groups with a fixed-point-free automorphism have been studied by several prominent group theorists; see for example [3,11].

3. Automorphisms of $(M, *)$

Let $(M,*)$ be an Alexander quandle. If $f \in \mathrm{Aut}(M,*)$, then $f'(x) = f(x) - f(0)$, $x \in M$, defines an automorphism $f' \in \mathrm{Aut}(M,*)$ with $f'(0) = 0$. Therefore it suffices to determine

$$\mathrm{Aut}_0(M,*) = \bigl\{f \in \mathrm{Aut}(M,*):\ f(0) = 0\bigr\}. \tag{3.1}$$

Let $\epsilon = 1 - t \in \Lambda$ and define $\mathrm{ann}_M(\epsilon) = \{x \in M:\ \epsilon x = 0\}$. The following theorem is an extension of Theorem 2.1 in [10]. The proof is essentially the same as that of [10].

Theorem 3.1. *Let $(M,*)$ and $(N,*)$ be two Alexander quandles.*

(i) *If $f : (M,*) \to (N,*)$ is an isomorphism such that $f(0) = 0$, then $f|_{\epsilon M} : \epsilon M \to \epsilon N$ is a Λ-module isomorphism.*
(ii) *If $|\mathrm{ann}_M(\epsilon)/\epsilon M \cap \mathrm{ann}_M(\epsilon)| = |\mathrm{ann}_N(\epsilon)/\epsilon N \cap \mathrm{ann}_N(\epsilon)|$ and $h : \epsilon M \to \epsilon N$ is a Λ-isomorphism, then h can be extended to a quandle isomorphism $f : (M,*) \to (N,*)$.*
(iii) *$(M,*) \cong (N,*)$ if and only if $\epsilon M \cong \epsilon N$ as Λ-modules and $|\mathrm{ann}_M(\epsilon)/\epsilon M \cap \mathrm{ann}_M(\epsilon)| = |\mathrm{ann}_N(\epsilon)/\epsilon N \cap \mathrm{ann}_N(\epsilon)|$.*

Proof. (i) We have

$$f\bigl((1-\epsilon)x + \epsilon y\bigr) = (1-\epsilon)f(x) + \epsilon f(y), \quad x, y \in M. \tag{3.2}$$

Letting $y = 0$ and $x = 0$, respectively, we have

$$f\bigl((1-\epsilon)x\bigr) = (1-\epsilon)f(x), \quad x \in M, \tag{3.3}$$

$$f(\epsilon y) = \epsilon f(y), \qquad y \in M. \tag{3.4}$$

Since $(1-\epsilon)M = M$, it follows from (3.2)–(3.4) that

$$f(x+\epsilon y) = f(x) + f(\epsilon y), \quad x, y \in M. \tag{3.5}$$

By (3.4), $f|_{\epsilon M}$ is an onto function from ϵM to ϵN. The function $f|_{\epsilon M}$ is also 1–1 since f is. By (3.3) and (3.5), $\underbrace{f|_{\epsilon M}}: \epsilon M \to \epsilon N$ is a Λ-map.

(ii) Let $\widetilde{h}: \epsilon M/\epsilon^2 M \to \epsilon N/\epsilon^2 N$ be the Λ-isomorphism induced by h. Consider the following diagram

$$\begin{tikzcd} M/\epsilon M \arrow[r, dashed, "g"] \arrow[d, "\mu_M"'] & N/\epsilon N \arrow[d, "\mu_N"] \\ \epsilon M/\epsilon^2 M \arrow[r, "\widetilde{h}"] & \epsilon N/\epsilon^2 N \end{tikzcd}$$

where both μ_M and μ_N are induced by multiplication by ϵ. Since

$$\begin{aligned} |\ker \mu_M| &= \bigl|\bigl(\epsilon M + \mathrm{ann}_M(\epsilon)\bigr)/\epsilon M\bigr| \\ &= \bigl|\mathrm{ann}_M(\epsilon)/\epsilon M \cap \mathrm{ann}_M(\epsilon)\bigr| \\ &= \bigl|\mathrm{ann}_N(\epsilon)/\epsilon N \cap \mathrm{ann}_N(\epsilon)\bigr| \\ &= |\ker \mu_N|, \end{aligned} \tag{3.6}$$

there exists a bijection $g: M/\epsilon M \to N/\epsilon N$ such that the above diagram commutes. We may further assume $g(0+\epsilon M) = 0+\epsilon N$. Let $C_M \subset M$ ($C_N \subset N$) be a system of representatives of the cosets in $M/\epsilon M$ ($N/\epsilon N$) such that $0 \in C_M$ ($0 \in C_N$). Then there exists a bijection $\widehat{g}: C_M \to C_N$ such that the following diagram commutes, where π_M (π_N) is the restriction of the canonical map $M \to M/\epsilon M$ ($N \to N/\epsilon N$).

$$\begin{tikzcd} C_M \arrow[r, "\widehat{g}"] \arrow[d, "\pi_M"'] & C_N \arrow[d, "\pi_N"] \\ M/\epsilon M \arrow[r, "g"] \arrow[d, "\mu_M"'] & N/\epsilon N \arrow[d, "\mu_N"] \\ \epsilon M/\epsilon^2 M \arrow[r, "\widetilde{h}"] & \epsilon N/\epsilon^2 N \end{tikzcd}$$

By the commutativity of the above diagram, for each $c \in C_M$,

$$h(\epsilon c) \equiv \epsilon \widehat{g}(c) \pmod{\epsilon^2 N}.$$

So there exists $l(c) \in \epsilon N$ such that $h(\epsilon c) = \epsilon \widehat{g}(c) + \epsilon l(c) = \epsilon(\widehat{g}(c) + l(c))$. Since $h(0) = 0$ and $\widehat{g}(0) = 0$, we can choose $l(0) = 0$. Now define

$$\begin{aligned} f: M &\longrightarrow N, \\ c + x &\longmapsto \widehat{g}(c) + l(c) + h(x), \quad c \in C_M,\ x \in \epsilon M. \end{aligned}$$

Then f is clearly a bijection and $f|_{\epsilon M} = h$. It remains to show that f is a quandle homomorphism, i.e.,

$$f\big((c_1 + x_1) * (c_2 + x_2)\big) = f(c_1 + x_1) * f(c_2 + x_2), \quad c_1, c_2 \in C_M,\ x_1, x_2 \in \epsilon M. \tag{3.7}$$

After expansion and cancellation, (3.7) becomes (is equivalent to)

$$h(\epsilon c_1) - \epsilon\big(\widehat{g}(c_1) + l(c_1)\big) = h(\epsilon c_2) - \epsilon\big(\widehat{g}(c_2) + l(c_2)\big). \tag{3.8}$$

Eq. (3.8) is true since both sides are equal to 0.

(iii) ($\Rightarrow$) Let $f : (M, *) \to (N, *)$ be a quandle isomorphism. By (i), $f|_{\epsilon M} : \epsilon M \to \epsilon N$ is a Λ-isomorphism. By (3.5), f induces a bijection (not necessarily a Λ-map) $\overline{f} : M/\epsilon M \to N/\epsilon N$. Also, $f|_{\epsilon M} : \epsilon M \to \epsilon N$ induces a Λ-isomorphism $\widetilde{f} : \epsilon M/\epsilon^2 M \to \epsilon N/\epsilon^2 N$. By (3.4), we have a commutative diagram

$$\begin{tikzcd} M/\epsilon M \arrow[r, "\overline{f}"] \arrow[d, "\mu_M"'] & N/\epsilon N \arrow[d, "\mu_N"] \\ \epsilon M/\epsilon^2 M \arrow[r, "\widetilde{f}"] & \epsilon N/\epsilon^2 N \end{tikzcd}$$

It follows that $|\ker \mu_M| = |\ker \mu_N|$; as we saw in (3.6), this implies that $|\mathrm{ann}_M(\epsilon)/\epsilon M \cap \mathrm{ann}_M(\epsilon)| = |\mathrm{ann}_N(\epsilon)/\epsilon N \cap \mathrm{ann}_N(\epsilon)|$.

($\Leftarrow$) This direction follows from (ii). $\square$

Remark. In Theorem 3.1(iii), when M and N are infinite, the condition $|\mathrm{ann}_M(\epsilon)/\epsilon M \cap \mathrm{ann}_M(\epsilon)| = |\mathrm{ann}_N(\epsilon)/\epsilon N \cap \mathrm{ann}_N(\epsilon)|$ cannot be replaced by the condition $|M| = |N|$. Let p be a prime. Let $M_1 = \mathbb{Z}_{p^2}$ and define $tx = (1-p)x$, $x \in \mathbb{Z}_{p^2}$. Then M_1 is a Λ-module. Note that $\epsilon M_1 = p\mathbb{Z}_{p^2} \cong \mathbb{Z}_p$ and t acts on ϵM_1 as the identity map. Similarly, let $N_1 = \mathbb{Z}_{p^3}$ and define $tx = (1-p^2)x$, $x \in \mathbb{Z}_{p^3}$. Then N_1 is a Λ-module, $\epsilon N_1 = p^2\mathbb{Z}_{p^3} \cong \mathbb{Z}_p$, and t acts on ϵN_1 as the identity map. So $\epsilon M_1 \cong \epsilon N_1$ as Λ-modules. However, $|\mathrm{ann}_{M_1}(\epsilon)/\epsilon M_1 \cap \mathrm{ann}_{M_1}(\epsilon)| = |p\mathbb{Z}_{p^2}/p\mathbb{Z}_{p^2}| = 1$ and $|\mathrm{ann}_{N_1}(\epsilon)/\epsilon N_1 \cap \mathrm{ann}_{N_1}(\epsilon)| = |p\mathbb{Z}_{p^3}/p^2\mathbb{Z}_{p^3}| = p$. Let A be an infinite abelian group and let $\alpha \in \mathrm{Aut}(A)$ such that $1-\alpha \in \mathrm{Aut}(A)$. Define $tx = \alpha x$, $x \in A$. Then A becomes a Λ-module. Let $M = M_1 \oplus A$ and $N = N_1 \oplus A$. Then $|M| = |A| = |N|$ but $|\mathrm{ann}_M(\epsilon)/\epsilon M \cap \mathrm{ann}_M(\epsilon)| = 1$, $|\mathrm{ann}_N(\epsilon)/\epsilon N \cap \mathrm{ann}_N(\epsilon)| = p$.

We now determine the elements of $\mathrm{Aut}_0(M, *)$. For each $f \in \mathrm{Aut}_0(M, *)$, let $f' \in \mathrm{Aut}_0(M, *)$ be an extension of $f|_{\epsilon M}$ given by Theorem 3.1(ii). Then $(f')^{-1} f \in \mathrm{Aut}_0(M, *)$ and $((f')^{-1} f)|_{\epsilon M} = \mathrm{id}$. Therefore it suffices to determine the elements of

$$G := \big\{ f \in \mathrm{Aut}(M, *) \colon\ f|_{\epsilon M} = \mathrm{id} \big\}. \tag{3.9}$$

We write $\mathrm{ann}(\epsilon) = \mathrm{ann}_M(\epsilon)$ and $K = \epsilon M \cap \mathrm{ann}(\epsilon)$. Let $D \subset M$ be a system of representatives of the cosets in $M/(\epsilon M + \mathrm{ann}(\epsilon))$ such that $0 \in D$. For each function $g : M/\epsilon M \to \mathrm{ann}(\epsilon)$, define

$$\begin{aligned} f_g : M &\longrightarrow M, \\ x &\longmapsto x + g(\bar{x}) \end{aligned} \tag{3.10}$$

where $\bar{x}$ is the image of x in $M/\epsilon M$. Moreover, for this function g and each $d \in D$, define

$$\begin{aligned} g_d : \operatorname{ann}(\epsilon)/K &\longrightarrow \operatorname{ann}(\epsilon)/K, \\ x+K &\longmapsto g(\overline{d+x})+K, \quad x \in \operatorname{ann}(\epsilon). \end{aligned} \tag{3.11}$$

Denote the identity map of $\operatorname{ann}(\epsilon)/K$ by I.

Theorem 3.2. *Elements of G are precisely f_g where $g : M/\epsilon M \to \operatorname{ann}(\epsilon)$ is a function satisfying the following conditions.*

(i) $g(\bar{0}) = 0$.
(ii) *For each $d \in D$, $I + g_d$ is a permutation of* $\operatorname{ann}(\epsilon)/K$.

Proof. 1° We assume that $g : M/\epsilon M \to \operatorname{ann}(\epsilon)$ satisfies (i) and (ii) and show that $f_g \in G$.

For $x, y \in M$, we have

$$f_g(x * y) = f_g\big((1-\epsilon)x + \epsilon y\big) = (1-\epsilon)x + \epsilon y + g(\bar{x})$$

and

$$\begin{aligned} f_g(x) * f_g(y) &= (1-\epsilon) f_g(x) + \epsilon f_g(y) \\ &= (1-\epsilon)\big(x + g(\bar{x})\big) + \epsilon\big(y + g(\bar{y})\big) \\ &= (1-\epsilon)x + g(\bar{x}) + \epsilon y. \end{aligned}$$

Thus $f_g(x * y) = f_g(x) * f_g(y)$, so f_g is an endomorphism of $(M, *)$.

Now we show that f_g is 1–1. Assume $x, y \in M$ such that $f_g(x) = f_g(y)$. Then $x - y = g(\bar{y}) - g(\bar{x}) \in \operatorname{ann}(\epsilon)$. There exists $d \in D$ such that $x, y \in d + \epsilon M + \operatorname{ann}(\epsilon)$. Write $x = d + x_1 + x_2$ and $y = d + y_1 + y_2$, where $x_1, y_1 \in \epsilon M$, $x_2, y_2 \in \operatorname{ann}(\epsilon)$. Since $x - y \in \operatorname{ann}(\epsilon)$, we have $x_1 - y_1 \in \operatorname{ann}(\epsilon)$, so $x_1 - y_1 \in K$. It follows from $f_g(x) = f_g(y)$ that

$$x_1 + x_2 + g(\overline{d+x_2}) = y_1 + y_2 + g(\overline{d+y_2}). \tag{3.12}$$

Therefore $x_2 + g(\overline{d+x_2}) \equiv y_2 + g(\overline{d+y_2}) \pmod K$, namely,

$$(I + g_d)(x_2 + K) = (I + g_d)(y_2 + K).$$

By (ii), we have $x_2 + K = y_2 + K$, i.e., $x_2 - y_2 \in K$. Then in (3.12) we have $\overline{d+x_2} = \overline{d+y_2}$, so $x_1 + x_2 = y_1 + y_2$. It follows that $x = y$.

Next we show that f_g is onto. Let $x \in M$. There exist $d \in D$ and $x_1 \in \epsilon M$ such that $x \equiv d + x_1 \pmod{\operatorname{ann}(\epsilon)}$. By (ii), there exists $x_2 \in \operatorname{ann}(\epsilon)$ such that $(I + g_d)(x_2 + K) = x - (d + x_1) + K$, i.e.,

$$x_2 + g(\overline{d+x_2}) \equiv x - (d + x_1) \pmod K.$$

Let $x_3 = x - (d + x_1) - (x_2 + g(\overline{d+x_2})) \in K$. Then

$$\begin{aligned} x &= d + x_1 + x_2 + x_3 + g(\overline{d+x_2}) \\ &= d + x_1 + x_2 + x_3 + g(\overline{d+x_1+x_2+x_3}) \\ &= f_g(x_1 + x_2 + x_3). \end{aligned}$$

2° We show that every $f \in G$ is of the form f_g for some function $g : M/\epsilon M \to \operatorname{ann}(\epsilon)$ satisfying (i) and (ii). Let $g(x) = f(x) - x$, $x \in M$. Then

$$\epsilon g(x) = \epsilon f(x) - \epsilon x = f(\epsilon x) - \epsilon x = \epsilon x - \epsilon x = 0.$$

Thus $g(x) \in \operatorname{ann}(\epsilon)$. Assume $x_1, x_2 \in M$ such that $x_1 - x_2 \in \epsilon M$. Then

$$\begin{aligned} g(x_1) - g(x_2) &= f(x_1) - f(x_2) - (x_1 - x_2) \\ &= f(x_2 + x_1 - x_2) - f(x_2) - (x_1 - x_2) \\ &= f(x_2) + f(x_1 - x_2) - f(x_2) - (x_1 - x_2) \\ &= f(x_1 - x_2) - (x_1 - x_2) \\ &= 0. \end{aligned}$$

So g can be treated as a function from $M/\epsilon M$ to $\operatorname{ann}(\epsilon)$. Since $f(0) = 0$, we have $g(\bar{0}) = 0$.

Let $d \in D$. We first show that $I + g_d$ is onto. Let $x \in \operatorname{ann}(\epsilon)$. There exists $y \in M$ such that $f_g(y) = d + x$, i.e., $y + g(\bar{y}) = d + x$. Then $y - d = x - g(\bar{y}) \in \operatorname{ann}(\epsilon)$. Write $y_1 = y - d$. Then $y_1 + g(\overline{d + y_1}) = y_1 + g(\bar{y}) = x$. Thus $(I + g_d)(y_1 + K) = x + K$. So $I + g_d$ is onto.

Next we show that $I + g_d$ is 1–1. Assume $(I + g_d)(x_1 + K) = (I + g_d)(x_2 + K)$, where $x_1, x_2 \in \operatorname{ann}(\epsilon)$. Then $x_1 + g(\overline{d + x_1}) \equiv x_2 + g(\overline{d + x_2}) \pmod K$. Put $w = x_1 + g(\overline{d + x_1}) - x_2 - g(\overline{d + x_2}) \in K$. We have

$$\begin{aligned} f_g(d + x_1) &= d + x_1 + g(\overline{d + x_1}) \\ &= d + x_2 + g(\overline{d + x_2}) + w \\ &= d + x_2 + w + g(\overline{d + x_2 + w}) \\ &= f_g(d + x_2 + w). \end{aligned}$$

Thus $d + x_1 = d + x_2 + w$, so $x_1 = x_2 + w$. Therefore $x_1 + K = x_2 + K$. $\square$

Let $\operatorname{Aut}_\Lambda(\epsilon M)$ denote the automorphism group of the Λ-module ϵM. Note that $\operatorname{Aut}_\Lambda(\epsilon M) = C_{\operatorname{Aut}(\epsilon M)}(t)$, the centralizer of t in the automorphism group of the abelian group ϵM. For each $\alpha \in \operatorname{Aut}_\Lambda(\epsilon M)$, fix an extension $\widehat{\alpha} \in \operatorname{Aut}(M, *)$ of α given by Theorem 3.1(ii). For each $a \in M$, let $r_a \in \operatorname{Aut}(M, *)$ be defined by $r_a(x) = x + a$, $x \in M$. The following corollary combines Theorem 3.2 and the earlier comments on the relations between the elements of $\operatorname{Aut}(M, *)$, $\operatorname{Aut}_0(M, *)$ and G.

Corollary 3.3. *We have*

$$\begin{aligned} \operatorname{Aut}(M, *) = \bigl\{ r_a \widehat{\alpha} f_g \colon\ & a \in M,\ \alpha \in \operatorname{Aut}_\Lambda(\epsilon M),\ g : M/\epsilon M \to \operatorname{ann}(\epsilon) \\ & \textit{satisfies (i) and (ii) in Theorem 3.2} \bigr\}. \end{aligned}$$

Moreover, the factorization $r_a \widehat{\alpha} f_g$ is unique.

We illustrate Theorem 3.2 using a concrete and simple example.

Example 3.4. Let $M = \mathbb{Z}_8$ considered as a Λ-module with the scalar multiplication defined by $tx = -3x$. We may write $t = -3$ (harmless abuse of notation). Then $\epsilon = 4$, $\operatorname{ann}(\epsilon) = 2\mathbb{Z}_8$, $K = 4\mathbb{Z}_8$, $M/\epsilon M = \mathbb{Z}_8/4\mathbb{Z}_8$, $M/(\epsilon M + \operatorname{ann}(\epsilon)) = \mathbb{Z}_8/2\mathbb{Z}_8$, $\operatorname{ann}(\epsilon)/K = 2\mathbb{Z}_8/4\mathbb{Z}_8$. Choose $D = \{0, 1\} \subset M$.

Let $g : M/\epsilon M = \mathbb{Z}_8/4\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} \to \operatorname{ann}(\epsilon) = 2\mathbb{Z} = \{0, 2, 4, 6\}$ be a function with $g(\bar{0}) = 0$. Then $I + g_0$ and $I + g_1$ are given by

$$I + g_0\colon \begin{cases} 0 + K \mapsto 0 + K, \\ 2 + K \mapsto 2 + g(\bar{2}) + K, \end{cases}$$

$$I + g_1\colon \begin{cases} 0 + K \mapsto g(\bar{1}) + K, \\ 2 + K \mapsto 2 + g(\bar{3}) + K. \end{cases}$$

So g satisfies (ii) of Theorem 3.2 if and only if $g(\bar{2}) \in \{0, 4\}$ and $g(\bar{1}), g(\bar{3}) \in \{0, 4\}$ or $g(\bar{1}), g(\bar{3}) \in \{2, 6\}$. For example, if g is given by

$$g\colon \begin{cases} \bar{0} \mapsto 0, \\ \bar{2} \mapsto 4, \\ \bar{1} \mapsto 2, \\ \bar{3} \mapsto 2, \end{cases}$$

the corresponding f_g in (3.10) is given by

$$f_g\colon \begin{cases} 0 \mapsto 0, & 4 \mapsto 4, \\ 1 \mapsto 3, & 5 \mapsto 7, \\ 2 \mapsto 6, & 6 \mapsto 2, \\ 3 \mapsto 5, & 7 \mapsto 1. \end{cases}$$

4. Structure of $\operatorname{Aut}(M, *)$

We maintain the notation in the previous section concerning an Alexander quandle $(M, *)$. The symmetric group of a set X is denoted by $\operatorname{Sym}(X)$.

Let $R = \{r_a\colon a \in M\}$, which is a subgroup of $\operatorname{Aut}(M, *)$ isomorphic to M. Every $f \in \operatorname{Aut}(M, *)$ has a unique factorization $f = r_a f'$, where $a \in M$ and $f' \in \operatorname{Aut}_0(M, *)$. Thus $\operatorname{Aut}(M, *) = R \cdot \operatorname{Aut}_0(M, *)$, where $R \cap \operatorname{Aut}_0(M, *) = \{\mathrm{id}\}$. Neither R nor $\operatorname{Aut}_0(M, *)$ is necessarily normal in $\operatorname{Aut}(M, *)$.

The group $\operatorname{Aut}_0(M, *)$ can be obtained from $K^{(M/\epsilon M)\setminus\{\bar{0}\}}$, the group of functions from $(M/\epsilon M) \setminus \{\bar{0}\}$ to K, through two consecutive extensions; the factor groups of the extensions are given in diagram:

$$\begin{tikzcd}
 & & & \operatorname{Aut}_0(M,*) \arrow[d, equal] & & \\
 & 1 \arrow[r] & G \arrow[r, "\subset"] \arrow[d, equal] & \operatorname{Aut}_0(M,*) \arrow[r, "\rho"] & \operatorname{Aut}_\Delta(\epsilon M) \arrow[r] & 1 \\
0 \arrow[r] & G_0 \arrow[r, "\subset"] \arrow[d, equal, "\wr"'] & G \arrow[r, "\Delta"] & \operatorname{im}\Delta \arrow[r] & 1 & \\
 & K^{(M/\epsilon M)\setminus\{\bar{0}\}} & & & &
\end{tikzcd} \tag{4.1}$$

In (4.1),

$$\operatorname{im}\Delta = \operatorname{Sym}\bigl(\bigl(\operatorname{ann}(\epsilon)/K\bigr) \setminus \{0\}\bigr) \times \prod_{d \in D\setminus\{0\}} \operatorname{Sym}\bigl(\operatorname{ann}(\epsilon)/K\bigr).$$

In what follows we describe the extensions in (4.1) in details. Define

$$\begin{aligned}\rho : \mathrm{Aut}_0(M,*) &\longrightarrow \mathrm{Aut}_\Lambda(\epsilon M),\\ f &\longmapsto f|_{\epsilon M}.\end{aligned}$$

By Theorem 3.1, ρ is an onto homomorphism. Clearly, $\ker\rho = G$.

Define

$$\begin{aligned}\Delta : G &\longrightarrow \mathrm{Sym}\big((\mathrm{ann}(\epsilon)/K)\setminus\{0\}\big)\times\prod_{d\in D\setminus\{0\}}\mathrm{Sym}\big(\mathrm{ann}(\epsilon)/K\big),\\ f_g &\longmapsto \big(I+g_0, (I+g_d)_{d\in D\setminus\{0\}}\big)\end{aligned}$$

where $g : M/\epsilon M \to \mathrm{ann}(\epsilon)$ satisfies Theorem 3.2 (i) and (ii). We claim that Δ is a homomorphism. Let $f_g, f_{g'} \in G$, where $g, g' : M/\epsilon M \to \mathrm{ann}(\epsilon)$ satisfy Theorem 3.2 (i) and (ii). Then

$$f_g f_{g'}(x) = x + g'(\bar{x}) + g\big(\bar{x} + \overline{g'}(\bar{x})\big), \quad x \in M,$$

where $\overline{(\,)}$ is the canonical homomorphism from M to $M/\epsilon M$ and $\overline{g'} = \overline{(\,)}\circ g'$. Therefore $f_g f_{g'} = f_{g''}$, where $g''(\bar{x}) = g'(\bar{x}) + g(\bar{x} + \overline{g'}(\bar{x}))$, $x \in M$. For $x \in \mathrm{ann}(\epsilon)$ and $d \in D$, we have

$$\begin{aligned}\big(I+g''_d\big)(x+K) &= x + g''(\overline{d+x}) + K\\ &= x + g'(\overline{d+x}) + g\big(\overline{d+x} + \overline{g'}(\overline{d+x})\big) + K.\end{aligned}$$

On the other hand,

$$\begin{aligned}(I+g_d)\big(I+g'_d\big)(x+K) &= (I+g_d)\big(x + g'(\overline{d+x}) + K\big)\\ &= x + g'(\overline{d+x}) + g\big(\overline{d+x} + \overline{g'}(\overline{d+x})\big) + K.\end{aligned}$$

Hence $I + g''_d = (I+g_d)(I+g'_d)$, so $\Delta(f_g f_{g'}) = \Delta(f_g)\Delta(f_{g'})$.

Next we show that Δ is onto. Let $(\sigma_0, (\sigma_d)_{d\in D\setminus\{0\}}) \in \mathrm{Sym}((\mathrm{ann}(\epsilon)/K)\setminus\{0\}) \times \prod_{d\in D\setminus\{0\}}\mathrm{Sym}(\mathrm{ann}(\epsilon)/K)$. We treat σ_0 as an element of $\mathrm{Sym}(\mathrm{ann}(\epsilon)/K)$ by setting $\sigma_0(0) = 0$. We try to define a function $g : M/\epsilon M \to \mathrm{ann}(\epsilon)$ satisfying Theorem 3.2 (i) and (ii). For each $x \in M$, write $x = d + x_1 + x_2$, where $d \in D$, $x_1 \in \epsilon M$, $x_2 \in \mathrm{ann}(\epsilon)$. Note that d is unique and x_2 is unique modulo K. If $\bar{x} \neq \bar{0}$, define $g(\bar{x})$ to be any element in $(\sigma_d - I)(x_2 + K)$; if $\bar{x} = \bar{0}$, define $g(\bar{0}) = 0$, which is of course an element of $0 + K = (\sigma_0 - I)(0 + K)$. For $d \in D$ and $x_2 \in \mathrm{ann}(\epsilon)$, we have

$$\begin{aligned}(I+g_d)(x_2+K) &= x_2 + g(\overline{d+x_2}) + K\\ &= x_2 + (\sigma_d - I)(x_2+K) \quad \text{(by the definition of } g)\\ &= \sigma_d(x_2+K).\end{aligned}$$

Thus $I + g_d = \sigma_d$, so Theorem 3.2(ii) is satisfied. Therefore $f_g \in G$ and obviously, $\Delta(f_g) = (\sigma_0, (\sigma_d)_{d\in D\setminus\{0\}})$. Let $G_0 = \ker\Delta$. Note that for a function $g : M/\epsilon M \to \mathrm{ann}(\epsilon)$ satisfying Theorem 3.2 (i) and (ii),

$$\begin{aligned}f_g \in G_0 \quad &\Leftrightarrow \quad g_d = 0 \quad \text{for all } d \in D\\ &\Leftrightarrow \quad \mathrm{im}\, g \subset K \quad \big(\text{by (3.11)}\big).\end{aligned}$$

Therefore,

$$G_0 = \{ f_g\colon\ g : M/\epsilon M \to K,\ g(\bar{0}) = 0 \}.$$

For $g, g' : M/\epsilon M \to K$ with $g(\bar{0}) = g'(\bar{0}) = 0$ and $x \in M$, we have

$$f_g f_{g'}(x) = f_g\big(x + g'(\bar{x})\big) = x + g'(\bar{x}) + g\big(\bar{x} + \overline{g'}(\bar{x})\big) = x + g'(\bar{x}) + g(\bar{x}) = f_{g+g'}(x).$$

So $f_g f_{g'} = f_{g+g'}$. Therefore the map

$$\begin{aligned} G_0 &\longrightarrow K^{(M/\epsilon M)\setminus\{\bar{0}\}}, \\ f_g &\longmapsto g \end{aligned}$$

is an isomorphism.

Theorem 4.1. *Let $(M, *)$ be a finite Alexander quandle. Then*

$$\big|\mathrm{Aut}(M, *)\big| = |\epsilon M||K|^{|M/\epsilon M|}\big(|\mathrm{ann}(\epsilon)/K|!\big)^{|M/(\epsilon M + \mathrm{ann}(\epsilon))|}\big|\mathrm{Aut}_\Lambda(\epsilon M)\big|. \tag{4.2}$$

Note. When M is infinite, (4.2) still holds as an equation of cardinals.

Proof of Theorem 4.1. We have

$$\begin{aligned} &\big|\mathrm{Aut}(M, *)\big| \\ &\quad = |M| \cdot \big|\mathrm{Aut}_0(M, *)\big| \\ &\quad = |M| \cdot |G_0| \cdot \left|\mathrm{Sym}\big(\big(\mathrm{ann}(\epsilon)/K\big) \setminus \{0\}\big) \times \prod_{d \in D\setminus\{0\}} \mathrm{Sym}\big(\mathrm{ann}(\epsilon)/K\big)\right| \cdot \big|\mathrm{Aut}_\Lambda(\epsilon M)\big| \quad \big(\text{by (4.1)}\big) \\ &\quad = |M||K|^{|M/\epsilon M|-1} \cdot \frac{1}{|\mathrm{ann}(\epsilon)/K|}\big(\big|\mathrm{ann}(\epsilon)/K\big|!\big)^{|D|}\big|\mathrm{Aut}_\Lambda(\epsilon M)\big| \\ &\quad = |\epsilon M||K|^{|M/\epsilon M|}\big(\big|\,\mathrm{ann}(\epsilon)/K\big|!\big)^{|M/(\epsilon M + \mathrm{ann}(\epsilon))|}\big|\mathrm{Aut}_\Lambda(\epsilon M)\big|. \qquad \square \end{aligned}$$

Example 4.2. Consider the Alexander quandle in Example 3.4: $M = \mathbb{Z}_8$, $t = -3$. In diagram (4.1) we have

$$\begin{aligned} &K^{(M/\epsilon M)\setminus\{\bar{0}\}} = (4\mathbb{Z}_8)^3 \cong \mathbb{Z}_2^3; \\ &\mathrm{im}\,\Delta = \mathfrak{S}_1 \times \mathfrak{S}_2 \cong \mathfrak{S}_2, \quad \text{where } \mathfrak{S}_n \text{ is the symmetric group on } n \text{ elements}; \\ &|G| = 2^3 \cdot 2 = 2^4; \\ &\mathrm{Aut}_\Lambda(M) = \{\mathrm{id}\}; \\ &\big|\mathrm{Aut}_0(M, *)\big| = |G| = 2^4. \end{aligned}$$

In Theorem 4.1 we have $|\mathrm{Aut}(M, *)| = 2^7$.

5. Examples

Eq. (4.2) expresses $|\mathrm{Aut}(M,*)|$ in several ingredients. According to Theorem 3.1, $|\mathrm{Aut}(M,*)|$ should be determined by the Λ-module ϵM and the cardinality $|\mathrm{ann}(\epsilon)/K|$. In fact, it is not difficult to see that all ingredients in (4.2) are determined by ϵM and $|\mathrm{ann}(\epsilon)/K|$. In this section we examine two examples of Alexander quandles; in both examples, the ingredients in (4.2) are made explicit.

Example 5.1. Let $M=\mathbb{Z}_n$, where $n>1$ is an integer. Let $a\in\mathbb{Z}_n^\times$ and turn M into a Λ-module by defining $tx=ax$, $x\in M$. Assume $n=p_1^{e_1}\cdots p_s^{e_s}$ where $p_1,\dots,p_s$ are distinct primes and $e_i\geqslant 1$. Let k_i be the p_i-adic order of $\epsilon=1-a$ in $\mathbb{Z}_n$, i.e., the smallest nonnegative integer k such that $\epsilon\in p_i^k\mathbb{Z}_n$. Then

$$\begin{cases} |\epsilon M|=\prod_i p_i^{e_i-k_i}, \\ |K|=\prod_i p_i^{\min\{k_i,e_i-k_i\}}, \\ \bigl|\mathrm{ann}(\epsilon)\bigr|=\prod_i p_i^{k_i}, \\ \bigl|\epsilon M+\mathrm{ann}(\epsilon)\bigr|=\prod_i p_i^{\max\{k_i,e_i-k_i\}}. \end{cases} \tag{5.1}$$

Since $\mathrm{Aut}(\epsilon M)\cong\mathrm{Aut}(\prod_i\mathbb{Z}_{p^{e_i-k_i}})$ is abelian, we have $\mathrm{Aut}_\Lambda(\epsilon M)=\mathrm{Aut}(\epsilon M)\cong\prod_i\mathbb{Z}^\times_{p^{e_i-k_i}}$. Hence

$$\bigl|\mathrm{Aut}_\Lambda(\epsilon M)\bigr|=\prod_i(p_i-1)^{\delta(e_i-k_i)}p_i^{\max\{0,e_i-k_i-1\}}, \tag{5.2}$$

where

$$\delta(k)=\begin{cases}1 & \text{if}\, k\geqslant 1,\\ 0 & \text{if}\, k<1.\end{cases}$$

Making substitutions (5.1) and (5.2) in (4.2), we have

$$\begin{aligned}\bigl|\mathrm{Aut}(M,*)\bigr|={}&\Bigl[\prod_i p_i^{\min\{k_i,e_i-k_i\}}\Bigr]^{\prod_i p_i^{k_i}}\\ &\cdot\Bigl[\Bigl(\prod_i p_i^{\max\{0,2k_i-e_i\}}\Bigr)!\Bigr]^{\prod_i p_i^{\min\{k_i,e_i-k_i\}}}\\ &\cdot\Bigl[\prod_i(p_i-1)^{\delta(e_i-k_i)}p_i^{\max\{e_i-k_i,2e_i-2k_i-1\}}\Bigr].\end{aligned} \tag{5.3}$$

Example 5.2. Let p be a prime, $n>0$, and $M=\mathbb{Z}_p^n$. Let $A\in\mathrm{GL}(n,\mathbb{Z}_p)$ and turn M into a Λ-module by defining $tx=Ax$, $x=[x_1,\dots,x_n]^T\in M$. Let the elementary divisors of A be

$$p_i^{\lambda_{i1}},\dots,p_i^{\lambda_{i,s_i}},\quad 0\leqslant i\leqslant k,$$

where p_0 $(= X - 1)$, $p_1, \dots, p_k$ are distinct irreducible polynomials in $\mathbb{Z}_p[X]$ and $\lambda_{i1} \geqslant \cdots \geqslant \lambda_{i,s_i} \geqslant 1$. Write $\boldsymbol{\lambda}_i = (\lambda_{i1}, \dots, \lambda_{i,s_i})$.

For each partition $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_s \geqslant 1$, let $l(\boldsymbol{\lambda}) = s$ and $m_i(\boldsymbol{\lambda}) = |\{j\colon \lambda_j = i\}|$, $i \geqslant 1$. Let $a_{\boldsymbol{\lambda}}(q)$ be the function of q defined in [8, II (1.6)]:

$$a_{\boldsymbol{\lambda}}(q) = q^{\sum_{j=1}^{s}(2j-1)\lambda_j} \prod_{i \geqslant 1} \big[\big(1 - q^{-1}\big)\big(1 - q^{-2}\big) \cdots \big(1 - q^{-m_i(\boldsymbol{\lambda})}\big)\big].$$

We have

$$\begin{cases} |\epsilon M| = p^{n - l(\boldsymbol{\lambda}_0)}, \\ |K| = p^{l(\boldsymbol{\lambda}_0) - m_1(\boldsymbol{\lambda}_0)}, \\ \big|\mathrm{ann}(\epsilon)\big| = p^{l(\boldsymbol{\lambda}_0)}, \\ \big|\epsilon M + \mathrm{ann}(\epsilon)\big| = p^{n - l(\boldsymbol{\lambda}_0) + m_1(\boldsymbol{\lambda}_0)}. \end{cases} \tag{5.4}$$

The Λ-module ϵM is isomorphic to $\mathbb{Z}_p^{n - l(\boldsymbol{\lambda}_0)}$ with $tx = Bx$, $x \in \mathbb{Z}_p^{n - l(\boldsymbol{\lambda}_0)}$, where $B \in \mathrm{GL}(n - l(\boldsymbol{\lambda}_0), \mathbb{Z}_p)$ has elementary divisors

$$p_0^{\lambda_{01} - 1}, \dots, p_0^{\lambda_{0,s_0'} - 1} \quad \big(s_0' = l(\boldsymbol{\lambda}_0) - m_1(\boldsymbol{\lambda}_0)\big),$$
$$p_i^{\lambda_{i1}}, \dots, p_i^{\lambda_{i,s_i}}, \quad 1 \leqslant i \leqslant k.$$

Define $\boldsymbol{\mu} = (\lambda_{01} - 1, \dots, \lambda_{0,s_0'} - 1)$. By [8, IV (2.7)] we have

$$\big|\mathrm{Aut}_\Lambda(\epsilon M)\big| = \big|C_{\mathrm{GL}(n - l(\boldsymbol{\lambda}_0), \mathbb{Z}_p)}(B)\big| = a_{\boldsymbol{\mu}}(p) \prod_{i=1}^{k} a_{\boldsymbol{\lambda}_i}\big(p^{\deg p_i}\big). \tag{5.5}$$

(The same formula with a different appearance can be found in [6].) Substitutions (5.4) and (5.5) in (4.2) yield

$$\big|\mathrm{Aut}(M, *)\big| = p^{n - l(\boldsymbol{\lambda}_0) + l(\boldsymbol{\mu}) p^{l(\boldsymbol{\lambda}_0)}} \big(p^{m_1(\boldsymbol{\lambda}_0)}!\big)^{p^{l(\boldsymbol{\mu})}} a_{\boldsymbol{\mu}}(p) \prod_{i=1}^{k} a_{\boldsymbol{\lambda}_i}\big(p^{\deg p_i}\big).$$

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